Intuitionistic Logic and Topos Theory

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1. Brouwer

Any scientific activity is governed by logic. The word 'logic' here usually refers to 'sound' reasoning, and the forms of reasoning which are sound may differ somewhat from one discipline to another. For example, the logic of ethics, dealing with statements about what actions should or should not be performed, and the logic governing reasoning involving probabilities, both differ from the one pure descriptive, factual reasoning. The latter logic is usually thought of as the logic which applies to mathematics, which, after all, is the prime example of a scientific discipline where statements are clear and unambiguous, hence either true or false.

This, however, tacitly assumes agreement on the nature of mathematical knowledge, on what it means to 'know' that a mathematical statement is true. It was convincingly shown by the Dutch mathematician L.E.J. Brouwer (1881-1966, see figure 1) that there is no unique unambiguous approach to mathematical truth. Brouwer developed a *constructive* foundation of mathematics, in which a mathematical statement is only viewed as established when constructions are given for all the objects asserted to exist by the statement. To see how this affects the logic, note that from this point of view, 'classical' rules of logic such as *Tertium non datur* ('p or not p') and proof by contradiction ('if the assumption that not p leads to a contradiction, then p') are no longer valid. This becomes particularly clear for existential statements. For Brouwer, a statement of the form 'there

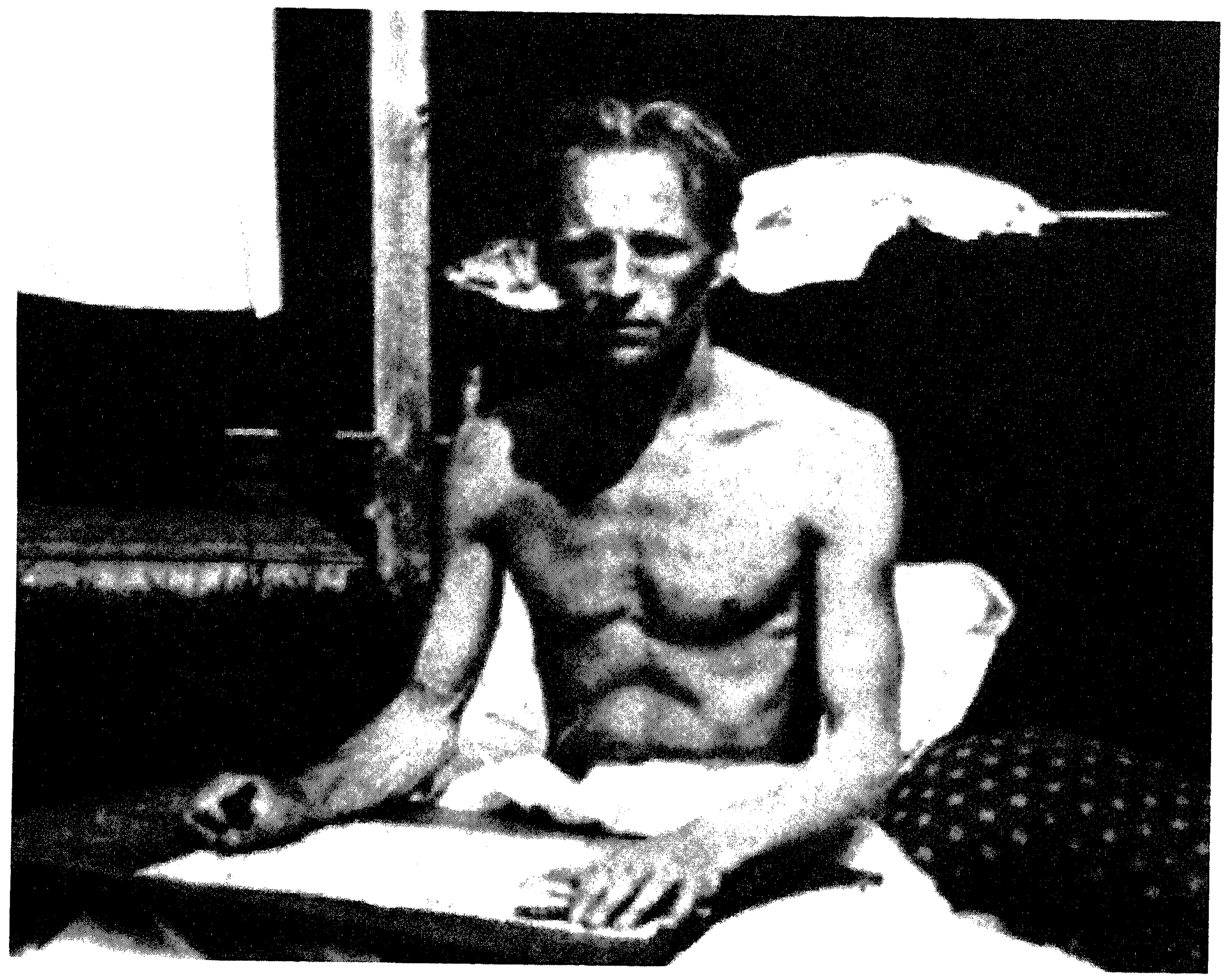

Figure 1. L.E.J. Brouwer (Photo: Brouwer archive).

exists an object x with property p' is established only when one can describe an explicit construction of such an object x; thus, it doesn't suffice to prove that the assumption that no object x can have property p leads to a contradiction.

Brouwer's ideas were later formalized and made into a clear logical system of axioms, called 'intuitionistic logic', first and notably by A. Heyting. This system, and variants of it, led to several interesting early developments, such as Gödel's embedding of intuitionistic logic into modal logic (around 1933), Kleene's algorithmic interpretation of intuitionistic logic (where the 'constructions' of Brouwer are interpreted as algorithms for numerical functions, around 1945), and Kripke's completeness proof (1965) using what are now called Kripke models.

It is often thought that intuitionistic logic, and the mathematics based on it, are properly contained in ordinary, 'classical' logic and mathematics. Now it is indeed true that intuitionistic logic per se is weaker than classical logic, but this also implies that there is room for new concepts and theorems, perhaps inconsistent with ordinary logic. For example, based on a suitable analysis of the notion of a real number, Brouwer argued that all functions from the reals to itself are continuous. This result was later made more rigorous by G. Kreisel and A.S. Troelstra, who extended Heyting's axioms to a system for analysis, containing so-called 'choice sequences'. In this system, Brouwer's continuity result is formally derivable.

Thus, intuitionistic logic turned out to be very rich in mathematical content, by giving rise to new methods and models of formal logic, and in the ways it deviates from and possibly extends classical mathematics. Nevertheless, it seems fair to say that it was not part of the mainstream activities in mathematics in general, and within mathematical logic in particular. This situation changed drastically in the past two decades, for two reasons. First, with the increasing interaction between logic and computer science, intuitionistic logic turned out to play a central role, in semantics of programs as well as in proof theory (program extraction from formal derivations). Secondly, the relation to sheaves and topoi, discovered in the early seventies, gave an enormous impulse to intuitionistic logic, and started a whole new and broader line of development, now generally refered to as categorical logic. It is this second reason that I wish to explore here.

2. GROTHENDIECK

The work of the French mathematician A. Grothendieck (see figure 2) in the sixties and seventies formed a revolution in algebraic geometry, and later led to the solution by P. Deligne of the famous Weil conjectures. Central among the many new concepts and methods introduced by Grothendieck was his generalization of the notion of 'space'. The basic idea was that for the construction of many invariants of a space, it suffices to know the system of all

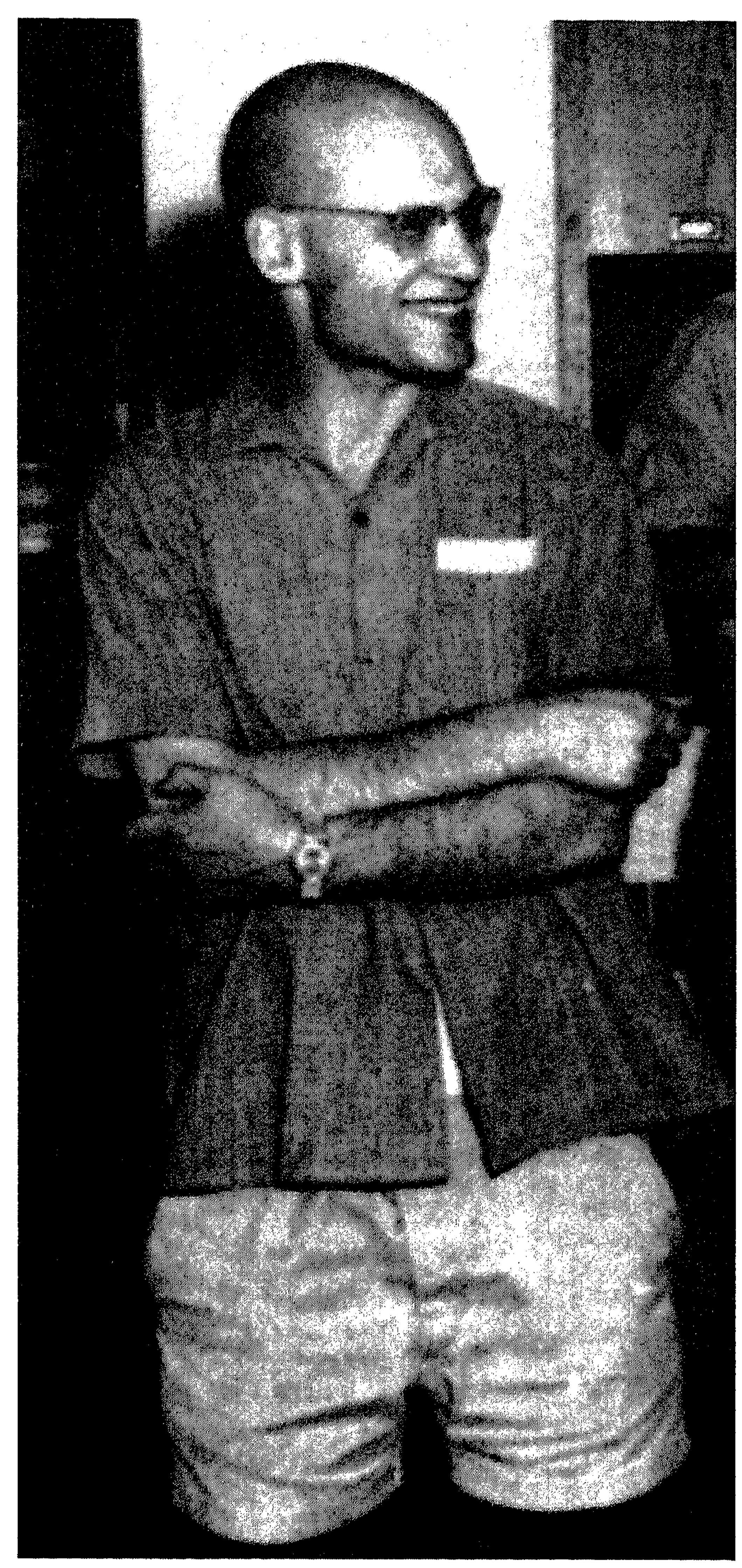

Figure 2. A. Grothendieck (Courtesy Birkhaüser, Inc., Boston).

'sheaves' which can be defined over the space. A sheaf is something like a continuously varying function on the space whose values are sets—or more often, sets with some algebraic structure, such as abelian groups. Grothendieck then observed that such sheaves could be defined not only for spaces, but for much more general structures. This gave rise to the notion of a 'site'. A site is a category, to be thought of as a system of 'neighbourhoods', equipped with an a priori given notion of when a family of such neighbourhoods covers another neighbourhood. For any such site one can define the system of all sheaves on the site. Such a system defined from a site is called a *topos*. According to Grothendieck topoi—and not just spaces—are the central geometric objects to be studied.

The relation to logic and set theory arose from the work of F.W. Lawvere and M. Tierney, concerned with simplifications of Grothendieck's axioms for sheaves and sites. Lawvere and Tierney discovered that many of the properties of topoi could be derived from a very simple set of axioms. These axioms describe elementary properties of sheaves: for example, that for any two sheaves S and T one can form the product sheaf $S \times T$, the 'function sheaf' T^S of all maps from S to T, and the 'powersheaf' PS of all subsheaves of S. They also discovered that any topos can be viewed as a 'universe of sets'. This means that one can interpret the axioms of set theory in a topos, and view this topos as a world in which one can do mathematics. Such a topos world is exactly like the ordinary world of sets in which mathematicians work, except that the logical rules of Tertium non datur and proof by contradiction do not hold. In fact, it is a world with a logic which differs from ordinary, classical logic: a world with precisely the intuitionistic logic of Brouwer and Heyting!

3. Topos models for intuitionistic logic

The discovery of this striking coincidence between geometric structures and intuitionistic logic immediately led to the construction of a great variety of natural mathematical models of specific intuitionistic theories. For example, the principle mentioned above of continuity of all real functions, at first thought of as a rare phenomenon, turned out to be true in many of Grothendieck's most general topoi (the so-called topological gros topoi). Using these topoi, one discovers natural models of the Kreisel-Troelstra theory of choice sequences (G.F. van der Hoeven and I. Moerdijk, 1984).

The use of topoi as models for logic and set theory also led to new explanations of classical independence results for Zermelo-Fraenkel set theory. Thus, Tierney showed (1972) how to interpret Cohen's famous proof of the independence of the Continuum Hypothesis as a topos theoretic construction, and later (1980) P. Freyd gave a strikingly simple proof, based on topos theory, of the independence of the axiom of choice. Around the same time, J.M.E. Hyland showed that Kleene's algorithmic interpretation of intuitionistic logic can also be viewed as the construction of a topos, thus providing an extension of Kleene's interpretation to higher order logic. Hyland's topos has various properties which make it strikingly different from ordinary set-theoretic universes. For example, it contains a large class of countable sets which has the remarkable property that the product of all its members is again a member of the class, while avoiding the paradoxes of set theory that such phenomena usually give rise to. This property has applications to the semantics of certain strong functional programming languages (versions of the so-called polymorphic lambda calculus).

The Continuum Hypothesis, formulated by G. Cantor in 1878, states that every infinite subset of the continuum **R** (i.e., the set of all real numbers) is either equivalent to the set of natural numbers or to **R** itself. D. Hilbert posed, in his celebrated list of problems presented at the International Congress of Mathematicians in 1900 in Paris, as Problem nr.1 that of proving this hypothesis. The independence of the Continuum Hypothesis, proven in 1963 by P.J. Cohen, means that it neither can be deduced from, nor contradicts the other axioms of set theory (e.g. the Zermelo-Fraenkel system), assuming these axioms to be non-contradictory. The Axiom of Choice (E. Zermelo, 1904) states that if $\mathbb{S}$ is a system of non-empty sets, then there exists a set A having exactly one element in common with every set S of $\mathbb{S}$. This axiom was put forward in connection with the question, posed by Cantor, whether of two sets there is always a largest. With the Axiom of Choice there is. The axiom met with considerable resistance, because it produced some counter-intuitive results. The independence in the above sense of the Axiom of Choice was also proved by Cohen. These independence results bear some similarity to the famous parallel postulate in Euclidean geometry, the discussion about which led to the discovery of non-Euclidean geometry. However, the independence proofs in logic are very different from those in geometry.

The newly discovered relation between intuitionistic logic and topos theory also led to an effective and well-motivated development of parts of intuitionistic mathematics. Notably, P.T. Johnstone and others developed a version of intuitionistic topology which avoids the use of points, and is now known as locale theory. This theory immediately went far beyond what was known up to then in intuitionistic topology. Furthermore, by applying the theory inside a topos (remember, a topos 'is' an intuitionistic universe in which one can do mathematics), various new presentation theorems for topoi were discovered. For example, Freyd proved that any topos allows a particularly nice embedding into the category of $\mathbb{Q}$-equivariant sheaves on a locale X equipped with the action by the group $\mathbb{Q}$ of rational numbers. A. Joyal proved with Tierney (1984) that any topos can be described as a more general category of equivariant sheaves on a locale. Later (1990), Joyal and Moerdijk proved that for any topos there exists a locale with the same weak homotopy type.

4. Further developments

The discovery of the relation between topoi and logic stood at the origin of an entirely new subject within logic, now called 'categorical logic'. In categorical logic, one tries to study logical systems in a way independent from their description in a specific language. Instead, a logical system is described by certain closure conditions on categories. The 'free' category possessing these closure conditions then replaces the older description of the logical system by a formal language, while more concrete, mathematical categories possessing these closure conditions correspond to models of the system. Semantics is now simply a functor between categories. (In hindsight, this is somewhat analogous to group theory, where abstract groups come instead of generators and relations, and representations of groups are homomorphisms into 'concrete' groups of automorphisms.) In categorical logic, the notion of a topos replaces that of a logical system for (a weak form of) set theory.

Similarly, logical systems for the typed lambda calculus correspond to cartesian closed categories [3]. First order logic corresponds to Grothendieck's theory of pretopoi (or coherent topoi). Grothendieck's fibered categories also have turned out to be very useful, in particular for describing type theories with so-called dependent types; see the Ph.D.-theses of D. Pavlovic (Utrecht, 1990) and B.P.F. Jacobs (Nijmegen, 1991).

This formulation of logic using categories has turned out to be very flexible and useful, in discovering analogies between logical systems, in finding new models, and, perhaps most importantly, in making methods of central parts of mathematics such as algebra and topology applicable to logic. By way of example, I may mention the work of M. Makkai (1993) and M. Zawadowski (1995) on descent theory. Descent theory in algebraic geometry is concerned with conditions under which one can 'descend' structures defined for a space X along a mapping $X \to Y$ to similar structures defined over Y. Descent theory was also used in the work of Joyal-Tierney and Joyal-Moerdijk mentioned above. The work of Makkai and Zawadowski relates descent theory, via the methods of categorical logic, to classical results in logic concerning definability and interpolation, some of which go back to the Dutch logician E.W. Beth (1909-1964).

Using the correspondence between suitable categories (pretopoi) and first order logic, Makkai and Zawadowski have been able to develop a descent theory for first order logic, and show that this theory leads to definability and interpolations theorems which are considerably stronger than the classical ones just mentioned.

This is one of many examples of an active and highly interesting interaction between logic and geometry, which has led and will lead to many new concepts and results in mathematical logic, and from which a fruitful feedback to geometry in general and topos theory in particular is emerging.

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